MAT 3004 – Abstract Algebra I

Tutorial 5

Group actions, formally We have, recurrently, emphasized that groups are actually collections of 'actions'. However we have never made this idea precise. We now formally define *group actions* to its maximum generality, and we will go through a number of examples.

Definition (Group action). For a group G and a set S, a *(left) group action* α of G on S is a function

$$\alpha : G \times S \to S$$

often denoted by $(g, s) \overset{\alpha}{\mapsto} g.s$ such that

a) $e.s = s$
b) $(gh).s = g.(h.s)$

for all $g, h \in G$ and $s \in S$.

Example 1. In all of the following examples, no matter what additional structure that S (the space being acted on) has, we 'forget' it: that is, we treat it as a mere set.

a) (Group acting on itself) It is almost trivial to verify that for all group G,

$$\ell : G \times G \to G, \quad (g, h) \overset{\ell}{\mapsto} gh$$

is a left group action. Show that

$$r : G \times G \to G, \quad (g, h) \overset{r}{\mapsto} hg^{-1}$$

and

$$\varphi : G \times G \to G, \quad (g, h) \overset{\varphi}{\mapsto} ghg^{-1}$$

are left group actions.

b) (General linear groups) The real general linear (or special linear) group $\mathrm{GL}(n, \mathbb{R})$ (or $\mathrm{SL}(n, \mathbb{R})$) acts on the vector space $\mathbb{R}^n$ (or $M_n(\mathbb{R})$) by $(A, x) \mapsto Ax$ where Ax is the usual matrix-vector product (or the usual matrix product).
c) (Symmetric groups) The symmetric group S_n (or the alternating group A_n) acts on the set $[n] = \{1, \cdots, n\}$ by $(\sigma, i) \mapsto \sigma(i)$ (recall the elements of S_n are bijections $\sigma : [n] \to [n]$)
d) (Automorphism groups) For any group G, the *automorphism group* $\mathrm{Aut}(G)$ is the group consisting of all isomorphisms $\phi : G \to G$. $\mathrm{Aut}(G)$ acts on G by $(\phi, g) \mapsto \phi(g)$.

Exercise E5.1 (What is not a group action?):
Show that the following are not group actions.

a) For any group G, is the following map

$$r^* : G \times G \to G, \quad (g, h) \overset{r^*}{\mapsto} hg$$

a left group action? What if G is an abelian group?

b) Consider the map that takes a pair $(A, X) \in M_n(\mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$ to $A + X$. Is it an action of $M_n(\mathbb{R})$ on $\mathrm{GL}(n, \mathbb{R})$?

Orbits and stabilizers Given a group action $\alpha : G \times S \to S$, we are interested in how a particular element $s \in S$ is affected by the group action. This motivates the following two definitions:

Definition (Orbit). The *orbit* of element s in S is the set of elements in S to which s can be 'moved', denoted by $G.s$:

$$G.s = \{g.s \mid g \in G\}$$

Two orbits are either disjoint or the same. In other words, the set of orbits of S under the action of G forms a partition of S. To see this, suppose $z \in G.x \cap G.y$, then there exists $h, k \in G$ s.t. $z = h.x = k.y$. This implies for all $g \in G$,

$$g.x = (gh^{-1}h).x = (gh^{-1}).(h.x) = (gh^{-1}).(k.y) = (gh^{-1}k).y \in G.y$$

Hence $G.x \subset G.y$. Using the same argument we have the reversed inclusion.

A group action is *transitive* if for all $s \in S$, the orbit $G.s$ of s is exactly S. By our previous argument, this is equivalent to: $\exists\, s \in S$ s.t. $G.s = S$.

Definition (Stabilizer). The *stabilizer subgroup* $\mathrm{Stab}(x)$ of G with respect to s is the set of all elements in G that fix x

$$\mathrm{Stab}(x) = \{g \in G \mid g.x = x\}$$

We haven't actually shown that $\mathrm{Stab}(x)$ is in fact a subgroup of G. But this is easy as soon as we notice the properties of a group action: for $g, h \in \mathrm{Stab}(x)$,

$$(gh).x = g.(h.x) = g.x = x, \quad \Rightarrow \quad gh \in \mathrm{Stab}(x)$$

and

$$x = e.x = (g^{-1}g).x = g^{-1}.(g.x) = g^{-1}.x, \quad \Rightarrow \quad g^{-1} \in \mathrm{Stab}(x)$$

Hence $\mathrm{Stab}(x)$ is a subgroup of G. Other names for $\mathrm{Stab}(x)$ include *isotropy group* and *little group.*

Exercise E5.2 (Orbits and stabilizers):
We re-examine some examples of group actions given above.

a) For any group G, show that ℓ and r defined in Example 1. a) are transitive group actions with stabilizer subgroups $\mathrm{Stab}(x)$ being trivial for all $x \in G$.
b) Consider the action $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ defined in Example 1. b). Find all orbits of this action.
c) For any group G and $H \triangleleft G$, consider a map from $G \times G/H$ to G/H by $(g, aH) \mapsto gaH$. Verify that this map defines a transitive action of G on G/H. Show that the stabilizer $\mathrm{Stab}(aH) = H$ for all $a \in G$.
d) Consider the action of S_n on $[n]$ defined in Example 1. c). Show that group action is transitive. Also show that the stabilizer subgroup $\mathrm{Stab}(i) \cong S_{n-1}$ for all $i \in [n]$.

We observe that in part (a) of Exercise E5.2, assuming G is finite, we have $|G| = |G| \cdot 1 = |G.x| \cdot |\mathrm{Stab}(x)|$; in part (c), assuming G is finite, we have by Lagrange's theorem that $|G| = |G/H| \cdot |H| = |G.aH| \cdot |\mathrm{Stab}(aH)|$; and in part (d), $|S_n| = n! = n \cdot (n-1)! = n \cdot |S_{n-1}| = |S_n.i| \cdot |\mathrm{Stab}(i)|$. Lagrange's theorem tells us in these cases we have

$$[G : \mathrm{Stab}(x)] = |G.x|$$

for all $x \in S$. Is this true only for the case of transitive actions?

Orbit-stabilizer theorem In this part, we provide a negative answer to the previous question: for ANY action of G on a set S, we have

$$[G : \mathrm{Stab}(x)] = |G.x|$$

for all $x \in S$. We now focus to prove this statement.

Let $G/\mathrm{Stab}(x)$ be the left coset space of $\mathrm{Stab}(x)$. We aim to establish a bijection between $G/\mathrm{Stab}(x)$ and $G.x$. We start by defining a map

$$\phi : G \to G.x, \quad \phi(g) = g.x$$

Clearly this map is a surjection by the definition of $G.x$. Now for some $g, h \in G$,

$$\phi(g) = \phi(h) \iff g.x = h.x \iff g^{-1}h.x = x \iff g^{-1}h \in \mathrm{Stab}(x) \iff g\mathrm{Stab}(x) = h\mathrm{Stab}(x)$$

Then ϕ induces a well-defined map

$$\tilde{\phi} : G/\mathrm{Stab}(x) \to G.x, \quad g\mathrm{Stab}(x) \mapsto G.x$$

By the previous argument, $\tilde{\phi}$ is a bijection as desired. Hence we conclude the proof.

When G is a finite group, Lagrange's theorem tells us $[G : \mathrm{Stab}(x)] = |G|/|\mathrm{Stab}(x)|$. Rearranging, the orbit-stabilizer theorem when G is finite takes the form

$$|G| = |G.x| \cdot |\mathrm{Stab}(x)|.$$

Exercise E5.3 (Conjugate class):
We now aim to apply the orbit-stabilizer theorem to solve some of the homework problems, related to conjugacy classes.

a) Conclude directly by the orbit-stabilizer theorem that in A_5, the 5-cycles cannot be in one orbit.

b) Recall the action φ of G on G defined in Example 1. a). Show that the conjugacy class of x

$$C_x := \{gxg^{-1} \mid g \in G\}$$

is the orbit $G.x$ of x. Show that the centralizer of x

$$Z_G(x) := \{g \in G \mid gxg^{-1} = x\}$$

Conclude directly that when G finite, $|C_x|$ divides $|G|$, and $|G|/|C_x| = |Z_G(x)|$.

More on conjugation In Homework 4, we have shown that the conjugation by g

$$\phi_g : G \to G, \quad x \mapsto gxg^{-1}$$

is an automorphism. These automorphisms have another name: *inner automorphisms*.

Exercise E5.4 (Inner automorphisms):
Let

$$\mathrm{Inn}(G) = \{\phi_g \mid g \in G\}$$

be the set of all inner automorphisms.

a) Show by the first isomorphism theorem that $G/Z(G) \cong \mathrm{Inn}(G)$, where $Z(G)$ is the center of G:

$$Z(G) = \{g \in G \mid gx = xg,\ \forall x \in G\}$$

(Hint: consider $\phi : G \to \mathrm{Aut}(G)$ by $g \mapsto \phi_g$. This also shows that $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$.)

b) Show that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

Roughly, $\mathrm{Inn}(G)$ can be seen as a measure of 'non-abelian-ness' - an inner automorphism ϕ_g is not the identity mapping implies g does not commute with all elements in G.

Since $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$, we can define the *outer automorphism group*, by

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$$

Then we have the following exact sequence:

$$1 \to Z(G) \to G \overset{\phi}{\to} \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

which means for any $G \overset{f_1}{\to} H \overset{f_2}{\to} K$ in the sequence, we have $\mathrm{im} f_1 = \ker f_2$.