Successively Solvable Shift-Add Systems — a Graphical Characterization

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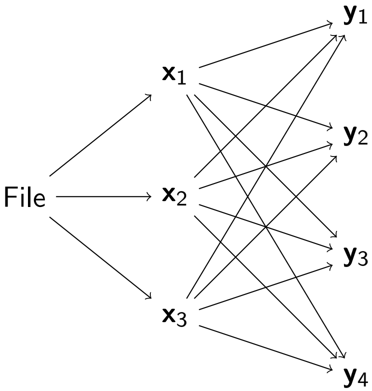
- Low complexity: only (non-cyclic) shift and XOR operations involved
- Applications:
  - Storage codes: Sung & Gong '13, Fu et al. '14, Dai et al. '17
  - Regenerating codes: Hou et al. '13, Fu et al. '15
  - Fountain codes: Nozaki '14, Jun et al. '17
  - Network codes: Sung & Gong '14

Toy example: storage coding



A storage system with 4 storage nodes that can tolerate one node failure.

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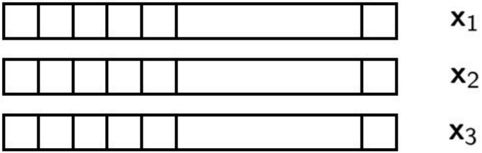

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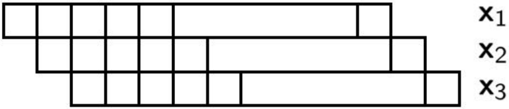

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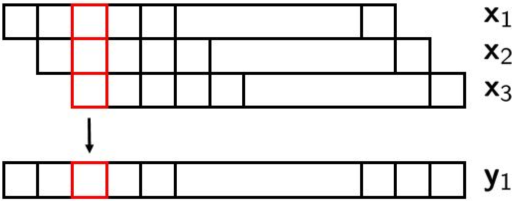

$$\mathbf{y}_1[2] = \mathbf{x}_1[2] + \mathbf{x}_2[1] + \mathbf{x}_3[0]$$

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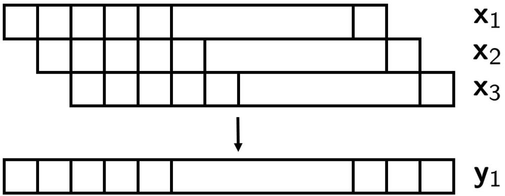

$$\mathbf{y}_1 = \mathbf{x}_1 + z\mathbf{x}_2 + z^2\mathbf{x}_3 = \begin{bmatrix} 1 & z & z^2 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix}^\top$$

where z^t is the shift operator, defined as

$$(z^t\mathbf{s})[\ell] = \begin{cases} \mathbf{s}[\ell - t], & \ell \geq \max\{t, 0\}, \\ 0, & \text{otherwise.} \end{cases}$$

- $t < 0$: discard/truncate the first $(-t)$ symbols
- $t = \infty$: $\mathbf{x}_i$ does not involve in the forming of $\mathbf{y}_1$

Shift-add system

- Model data symbols as elements of a finite abelian group $(\mathcal{A}, +)$. A sequence **s** is a mapping $\mathbb{Z} \to \mathcal{A}$ satisfying $\mathbf{s}[i] = 0$ for $i < 0$.

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- The output sequences are obtained by

$$\mathbf{y}_i = \sum_{j=1}^{n} z^{a_{ij}} \mathbf{x}_j$$

where the addition is performed in the additive group $\mathcal{A}$, and the shift operator z^t is defined as in the last slide:

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- The matrices $\Phi = (z^{a_{ij}})$ and $A = (a_{ij})$ are both called *shift matrices*.

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- An *infinite linear system* takes sequence $\mathbf{v}$ as input and outputs sequence $\mathbf{u}$, with

$$\mathbf{u}[j] = \sum_{i \in E_j} \mathbf{v}[i], \; \forall j \in \mathbb{N}_0$$

  where $\mathbb{N}_0 = \{n \in \mathbb{Z}, n \geq 0\}$, and $E_j \subseteq N_0$ is some finite index set for each $j \in \mathbb{N}_0$.

Infinite linear system: example

Example 1 (Shift-add system is an infinite linear system).

- Recall a shift-add system takes the form

$$\mathbf{y}_i = \sum_{j=1}^{n} z^{a_{ij}} \mathbf{x}_j \quad \text{or} \quad \mathbf{y}_i[\ell] = \sum_{j=1}^{n} \mathbf{x}_j[\ell - a_{ij}] = \sum_{j \in [n],\, \ell \geq a_{ij}} \mathbf{x}_j[\ell - a_{ij}]$$

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- Rearrange the entries in the input sequences $\mathbf{x}_j$ (resp. output sequences $\mathbf{y}_i$) into a single sequence $\mathbf{v}$ (resp. $\mathbf{u}$)., given by

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- Then the shift-add system can be rewritten in the notation of an infinite linear system as

$$\mathbf{u}[m\ell + i - 1] = \sum_{k \in E_{m\ell+i-1}} \mathbf{v}[k], \quad E_{m\ell+i-1} = \mathbb{N}_0 \cap \bigcup_{i=1}^{n} \{n(\ell - a_{ij}) + j - 1\}$$

Solvability of an infinite linear system

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Definition 1 (Zigzag solvable).

An infinite linear system is *zigzag solvable* if there exists two functions $f, g : \mathbb{N}_0 \to \mathbb{N}_0$ such that

1. f is bijective;
2. $\{f(i)\} \subseteq E_{g(i)} \subseteq \bigcup_{\ell=0}^{i} \{f(i)\}$, for all $i \in \mathbb{N}_0$.

Zigzag solvability: intuition

- Zigzag solvability ensures a solving algorithm called *zigzag decoding*.
- In each of the i-th stage ($i \in \mathbb{N}_0$), we use a symbol in sequence $\mathbf{u}$ to solve for a symbol in sequence $\mathbf{v}$.
- $f(i)$ indicates the index of the symbol in $\mathbf{v}$ solved in the i-th stage.
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- $g(i)$ indicates the index of the symbol in $\mathbf{u}$ used in the i-th stage.

- Condition 1 guarantees that every symbol in $\mathbf{v}$ is solved exactly once.
- Condition 2 guarantees that the substitution algorithm can proceed, i.e., $\mathbf{u}[g(i)]$ can be written as the sum of $\mathbf{v}[f(i)]$ and input symbols that are solved prior to the i-th stage.

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- Abusing notation: use (j, ℓ) in place of $(n\ell + j - 1)$ for the index of $\mathbf{x}_j[\ell]$ in $\mathbf{v}$, and (i, ℓ) in place of $(m\ell + i - 1)$ for the index of $\mathbf{y}_i[\ell]$ in $\mathbf{u}$.

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Definition 2 (Successively solvable).

A shift-add system is *successively solvable* if it is zigzag solvable with functions f, g, and satisfies an additional condition:

3. $f^{-1}(j, \ell) < f^{-1}(j, \ell')$ whenever $0 \leq \ell < \ell'$, for all $1 \leq j \leq n$.

Successive solvability: intuition

- Successive solvability guarantees that the symbols of each variable sequence are solved **successively** from left to right.
- This adds regularity to the solving algorithm, as in each stage, the function f takes value in the index of the first unsolved symbol of each variable sequence.

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- This adds regularity to the solving algorithm, as in each stage, the function f takes value in the index of the first unsolved symbol of each variable sequence.
- For shift-add systems, zigzag solvable does NOT implies successively solvable.

Zigzag solvability and successive solvability

Example 2 (Zigzag solvable $\not\Rightarrow$ successively solvable).

- Consider the shift-add system with shift matrix

$$A = \begin{pmatrix} 0 & 0 & \infty \\ \infty & -1 & \infty \\ -1 & 0 & 1 \end{pmatrix}$$

- The system is zigzag solvable but not successively solvable.
- In the 0-th stage, can only decode $\mathbf{x}_2[\ell]$ for some $\ell \geq 1$.
- $f^{-1}(2,0) > f^{-1}(2,\ell)$, violating successive solvability.

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Proposition 1.

For a shift-add system corresponding to a **non-negative** *shift matrix, it is zigzag solvable if and only if it is successively solvable.*

Reduction of shift-add systems

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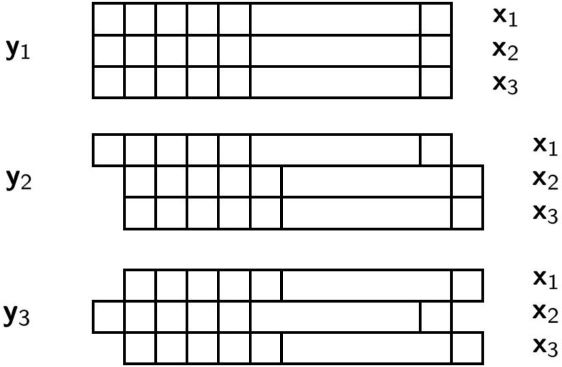
- A decoding stage can be seen as a reduction of a shift-add system.
- Let $\mathbf{x}'_j = z^{-1}\mathbf{x}_j$, i.e., $\mathbf{x}'_j[t] = \mathbf{x}_j[t+1]$ for $t \geq 0$.
- Let $\mathbf{y}'_i = z^{-1}\mathbf{y}_i$, i.e., $\mathbf{y}'_i[t] = \mathbf{y}_i[t+1]$ for $t \geq 0$.
- For $1 \leq k \leq n$, $k \neq i$, let $\mathbf{y}'_k$ be such that

$$\mathbf{y}'_k[t] = \begin{cases} \mathbf{y}_k[t] - \mathbf{x}_j[0], & \text{if } t = a_{kj}, \\ \mathbf{y}_k[t], & \text{otherwise,} \end{cases} \qquad \text{for } t \geq 0.$$

Reduction: example



Shift matrix: $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

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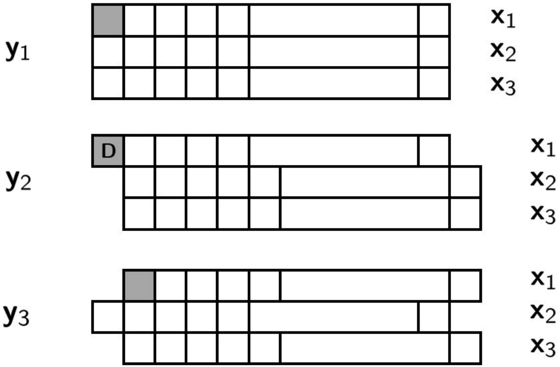

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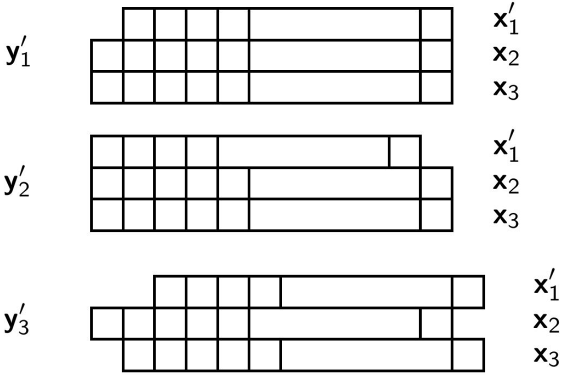

Shift matrix: $A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

Definitions on shift matrix

Definition 3 (Pivot, reductive).

Given a shift matrix A, the (i, j)-th entry is called a *pivot* if (i) $a_{ij} \geq 0$, and (ii) $a_{ij} < a_{ik}$ for all $k \neq j$. We say that a shift matrix A is *reductive* if A contains a pivot.

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Definition 4 (Equivalent).

Denote the i-th row of matrix A by $\mathbf{a}_i$ and the all-one row vector by $\mathbf{1}$. Two shift matrices A and B are *equivalent*, or $A \sim B$, if $\mathbf{a}_i - \mathbf{b}_i = c_i \cdot \mathbf{1}$, where $c_i = 0$ whenever $\exists\, j$ s.t. $a_{ij} < 0$ or $b_{ij} < 0$.

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Definition 5 (Reducing operator).

The (i,j)-th *reducing operator* R_{ij} is defined by $(R_{ij})_{k\ell} = \delta_{j,\ell} - \delta_{i,k}$, where $\delta_{\cdot,\cdot}$ is the Kronecker's delta function.

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Lemma 2.

Suppose the (i,j)-th entry of an $n \times n$ shift matrix A is a pivot. If $A' \sim A$, then the (i,j)-th entry is a pivot of A', and $(R_{ij} + A) \sim (R_{ij} + A')$.

Definitions on $\mathcal{G}_n$

Definition 6 ($\mathcal{G}_n$).

For a fixed positive integer n, we define a directed (multi)graph $\mathcal{G}_n$ whose vertices are equivalence classes of shift matrices with finite entries. Given two shift matrices A and B with finite entries, there is a directed edge from $\{A\}$ to $\{B\}$ (written $\{A\} \xrightarrow{R_{ij}} \{B\}$) if A has a pivot (i,j) and $R_{ij} + A \sim B$.

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Definition 7 (Path, cycle).

For positive integer L, we define a *path* of length L in $\mathcal{G}_n$ as a sequence of edges $\{A_k\} \xrightarrow{R_{i_k j_k}} \{B_k\}$, $k = 1, 2, \cdots, L$, where $B_k \sim A_{k+1}$ for $k = 1, 2, \cdots, L-1$. When $B_L \sim A_1$, the path is called a *cycle*.

Main result

Theorem 8.

An $n \times n$ shift-add system defined by shift matrix A with only finite entries is successively solvable if and only if there exists $\{B\}$ in a cycle of $\mathcal{G}_n$ and there exists a path from $\{A\}$ to $\{B\}$.

Main result

Theorem 8.

An $n \times n$ shift-add system defined by shift matrix A with only finite entries is successively solvable if and only if there exists $\{B\}$ in a cycle of $\mathcal{G}_n$ and there exists a path from $\{A\}$ to $\{B\}$.

Intuition: if a shift-add system is solvable, the infinite sequence of reducing operators ends up repeating a recurrent pattern.

Future work

- Infinity entries?
- Rectangular matrices?
- Algorithm for successive solvability?

Future work

Thank you!